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THE STATE OF STRESS OF AN ELASTICALLY SUPPORTED TRANSVERSELY ISOTROPIC BEAM†

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The bending of a three-layer beam with stiff outer layers is studied. The middle layer (spacer) has small stiffness compared with the outer layers and can be regarded as an elastic Winkler–Zimmerman support. By assumption, the lower (stiffest) isotropic layer behaves as a Bernoulli–Euler beam if a load is applied. The boundary conditions at the ends of the beam can be arbitrary, in general. The case when the ends of the lower beam are fixed is considered. A uniformly distributed compressing load acts on the outer surface of the upper transversely isotropic layer. The ends of this layer are free of any loads. The plane state of stress of the transversely isotropic layer is determined by the equations of elasticity theory. The exact solution of the problem is obtained in terms of trigonometric series.

IN THE general approach to the analysis of multi-layer constructions with soft and stiff layers [1, 2], the stiff layers are usually modelled using the Kirchhoff–Love hypothesis. As for the soft layers, it is assumed, for example, that all the components of the displacement vector are linear functions across the layer.

We shall solve the problem in terms of dimensionless coordinates x and y relative to the half-thickness l of the beam. The x-axis is parallel to the beam, while the y-axis is perpendicular to the transversely isotropic layer. The surfaces $x=\pm 1$ represent the ends of the beam, the contact and outer lateral surfaces of the transversely isotropic layer being given by $y=0$ and $y=h$, respectively.

The equations of equilibrium and consistency for the strains have the form

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,\quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0,\quad \frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} \tag{1}$$

We will write the relations between the stresses and strains ε_x, ε_y and $\frac{1}{2}\varepsilon_{xy}$ in the form

$$E_x\varepsilon_x=\sigma_x-k\nu'\sigma_y,\quad E_x\varepsilon_y=k\sigma_y-k\nu'\sigma_x,\quad E_x\varepsilon_{xy}=\gamma\sigma_{xy} \tag{2}$$
$$k=E_x/E_y,\quad \gamma=E_x/G$$

The y-axis is the axis of symmetry of the material, E_x and E_y are the moduli of elasticity in the x and y directions, G is the shear modulus, and ν' is Poisson's ratio.

The differential equation for the curved axis of the lower beam has the form

$$\frac{d^2W_0}{dx^2}=\frac{12}{E_0s_0^3}\left(\int_0^x\int_0^\eta R(\xi)d\xi d\eta-C\right) \tag{3}$$

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The transverse deformation of the spacer can be described by the equation

$$E_x(W(0,x) - W_0(x)) = \lambda_0 R(x), \quad R(x) = \sigma_y(0,x) \tag{4}$$

where $W(y, x)$ and $W_0(x)$ are dimensionless deflections (relative to l) of the transversely isotropic layer and the lower beam.

To compute λ_0 one can use the relation

$$\lambda_0 = E_x s_m / E_m, \quad E_m << E_x$$

where E_0 and E_m are the moduli of elasticity and s_0 and s_m are the thicknesses of the beam and the spacer, respectively.

In (3) and (4) the normal stress $R(x)$ is, of course, assumed to be constant across the spacer.

We will write the boundary conditions on the lateral surfaces $y=0$ (the contact surface) and $y=h$ (the outer surface) of the transversely isotropic layer as well as the ends of $x=\pm 1$ of the layer

$$y = 0\colon \sigma_y = R(x), \quad y = h\colon \sigma_y = -q \tag{5}$$

$$y = 0, \ y = h\colon \ \sigma_{xy} = 0$$

$$x = \pm 1\colon \int_0^h \sigma_x dy = 0, \quad \int_0^h \sigma_x y dy = 0, \quad \int_0^h \sigma_{xy} dy = 0 \tag{6}$$

In a more rigid form, the latter condition is $\sigma_{xy} = 0$. The undefined constant C in (3) can be found from the support conditions for the Bernoulli–Euler beam

$$x = \pm 1\colon \ dW_0/dx = 0, \ W_0 = 0. \tag{7}$$

The general integral of the third equation in (1) has the form

$$\sigma_x = \Phi(x) + F(x)y - \mu\sigma_y - k\int_0^y \int_0^\eta (\sigma_y)''_{xx} d\xi d\eta \tag{8}$$

$$\mu = \gamma - 2k\nu', \quad (\sigma_y)''_{xx} = \partial^2 \sigma_y / \partial x^2$$

where $\Phi(x)$ and $F(x)$ are undefined functions.

It follows from (3) and (4) (taking into account that $(W)''_{xx} \equiv (\varepsilon_{xy})'_x - (\varepsilon_x)'_y$) that

$$\lambda_0 R'' = -F - \left(\int_0^x \int_0^\eta R d\xi d\eta - C \right) 12 E_x / (E_0 s_0^3) \tag{9}$$

We require that the first condition in (6) be satisfied for every x (thus $\Phi(x)$ is also defined).

This being the case, if $\sigma_{xy}(0, x) = 0$, then the condition $\sigma_{xy}(h, x) = 0$ is satisfied automatically.

If the equilibrium equations and the condition $\sigma_{xy}(0, x)$ are taken into account, the second boundary condition in (5) takes the form

$$\lambda R^{(4)} + \omega R = \mu L_1((\sigma_y)''_{xx}) + (k/\lambda) L_2(\lambda(\sigma_y)''''_{xxxx} + \omega\sigma_y) - (k\omega/\lambda) L_2(\sigma_y) - q \tag{10}$$

$$\lambda = \lambda_0 h^3 / 12, \quad \omega = 1 + E_x h^3 / (E_0 s_0^3)$$

$$L_1(\cdot) = \int_0^h \int_0^y (\cdot) d\xi dy - \frac{h}{2} \int_0^h (\cdot) dy$$

$$L_2(\cdot) = \int_0^h \int_0^y \int_0^\xi \int_0^\eta (\cdot) d\rho d\eta d\xi dy - \frac{h}{2} \int_0^h \int_0^y \int_0^\xi (\cdot) d\eta d\xi dy$$

(L_1 and L_2 are operators acting on the y coordinate).

The contact stress $\sigma_y(0, x) = R(x)$ can be determined from Eq. (10), the exact solution of which can be found in terms of trigonometric series. To this end one must use the solutions of (1) in terms of Fourier series satisfying only the boundary conditions (5) on the lateral surfaces $y=0$ and $y=h$ of the layer.

We will assume that

$$R(x) = \sum_{n=0}^{\infty} a_n \cos \pi n x, \quad R''(x) = \sum_{n=0}^{\infty} b_n \cos \pi n x \tag{11}$$

We observe that, due to the symmetry of the problem, the third condition in (6) implies that

$$\int_0^1 R(x)dx = -q.$$

Hence we find that

$$a_0 = -q \tag{12}$$

Given the boundary conditions (6) on the surfaces $y=0$ and $y=h$ and relations (11) and (12), the solution of system (1) has the form (M being an undefined constant)

$$\begin{aligned}
\sigma_y &= \sum_{n=1}^{\infty} a_n Y_n(y) \cos \pi n x + a_0 \\
\sigma_{xy} &= -\sum_{n=1}^{\infty} a_n X_n(y) \sin \pi n x \\
\sigma_x &= -\sum_{n=1}^{\infty} a_n Z_n(y) \cos \pi n x + M(2y-h)
\end{aligned} \tag{13}$$

$$Y_n(y) = C_1 \operatorname{ch}(\alpha_{1,n} y) + C_2 \operatorname{sh}(\alpha_{1,n} y) + C_3 \operatorname{ch}(\alpha_{2,n} y) + C_4 \operatorname{sh}(\alpha_{2,n} y)$$

$$X_n(y) = \kappa_1 [C_1 \operatorname{sh}(\alpha_{1,n} y) + C_2 \operatorname{ch}(\alpha_{1,n} y)] + \kappa_2 [C_3 \operatorname{sh}(\alpha_{2,n} y) + C_4 \operatorname{ch}(\alpha_{2,n} y)]$$

$$Z_n(y) = \kappa_1^2 [C_1 \operatorname{ch}(\alpha_{1,n} y) + C_2 \operatorname{sh}(\alpha_{1,n} y)] + \kappa_2^2 [C_3 \operatorname{ch}(\alpha_{2,n} y) + C_4 \operatorname{sh}(\alpha_{2,n} y)]$$

$$\kappa_{\frac{1}{2}} = \{\tfrac{1}{2}[\mu \pm (\mu^2 - 4k)^{1/2}]\}^{1/2}, \quad \alpha_{1,n} = \pi n \kappa_1, \quad \alpha_{2,n} = \pi n \kappa_2, \quad \mu > 2\sqrt{k}$$

The undefined constants $C_1, \dots, C_4$ can be found from the system of equations

$$Y_n(0) = 1, \quad X_n(0) = 0, \quad Y_n(h) = 0, \quad X_n(h) = 0$$

for each n ($n \geqslant 1$)

The stresses σ_y and σ_{xy} can be uniquely defined from the boundary conditions (5) at the surfaces $y=0$ and $y=h$. We also note that condition (12) means that $\sigma_{xy}(y, \pm 1) = 0$.

The stress σ_x is written down apart from the term $M(2y-h)$, where

$$\int_0^h \sigma_x dy = 0$$

for any x.

The function $(\sigma_y)''_{xx}$ is uniquely defined by the boundary conditions $(\sigma_{xy})''_{xx} = 0$ for $y=0$ and $y=h$, $(\sigma_y)''_{xx} = 0$ for $y=h$, and $(\sigma_y)''_{xx} = R''(x)$ for $y=0$, and the system of equations (1) (the equations must be differentiated twice with respect to x)

$$(\sigma_y)''_{xx} = \sum_{n=1}^{\infty} b_n Y_n(y) \cos \pi n x + b_0 \{6h^{-3}(y^3/3 - y^2 h/2) + 1\} \tag{14}$$

As a result, we obtain

$$L_1((\sigma_y)''_{xx}) = \sum_{n=1}^{\infty} l_n^1 b_n \cos\pi n x + l_0^1 b_0$$
$$L_2(\sigma_y) = \sum_{n=1}^{\infty} l_n^2 a_n \cos\pi n x + l_{0,0}^2 a_0 \qquad (15)$$
$$L_2((\sigma_y)''_{xx}) = \sum_{n=1}^{\infty} l_n^2 b_n \cos\pi n x + l_{2,0}^2 b_0$$

where the constants l_n^1, l_n^2, $(n \geqslant 1)$, l_0^1, $l_{0,0}^2$, and $l_{2,0}^2$ are known. In particular, $l_0^1 = h^2/10$, $l_{0,0}^2 = -h^4/4!$, and $l_{2,0}^2 = -h^4 26/(7\times 5!)$.

In (10) we represent $L_1((\sigma_y))''_{xx}$ and $L_2(\sigma_y)$ by the Fourier series (15). Then the general solution of (10) has the form

$$R(x) = D_1 \operatorname{ch}\beta x \cos\beta x + D_2 \operatorname{sh}\beta x \sin\beta x +$$
$$+\mu\left\{\sum_{n=1}^{\infty} l_n^1 b_n [\lambda(\pi n)^4 + \omega]^{-1}\cos\pi n x + l_0^1 \frac{b_0}{\omega}\right\} - \qquad (16)$$
$$-\frac{k\omega}{\lambda}\left\{\sum_{n=1}^{\infty} l_n^2 a_n [\lambda(\pi n)^4 + \omega]^{-1}\cos\pi n x + l_{0,0}^2 \frac{a_0}{\omega}\right\} + \frac{k}{\lambda} L_2(\sigma_y) - \frac{q}{\omega}$$

where $\beta = [\omega/(4\lambda)]^{1/4}$.

We consider the series

$$\sum_{n=0}^{\infty} u_n(x), \quad u_n(x) = l_n^1 b_n [\lambda(\pi n)^4 + \omega]^{-1}\cos\pi n x$$

We assume that the original series

$$\sum_{n=0}^{\infty} l_n^1 b_n \cos\pi n x$$

converges. Then the series $\Sigma_{n=0}^{\infty} u_n'$ and $\Sigma_{n=0}^{\infty} u_n''$ formed by the derivatives are uniformly convergent. It follows that the series under consideration can be differentiated twice term-by-term [3]. This is also the case for the second series in formula (16).

We have

$$R'' = -2\beta^2 \operatorname{sh}\beta x \sin\beta x D_1 + 2\beta^2 \operatorname{ch}\beta x \cos\beta x D_2 -$$
$$-\mu \sum_{n=1}^{\infty} l_n^1 b_n (\pi n)^2 [\lambda(\pi n)^4 + \omega]^{-1}\cos\pi n x + \qquad (17)$$
$$+\frac{k\omega}{\lambda}\sum_{n=1}^{\infty} l_n^2 a_n (\pi n)^2 [\lambda(\pi n)^4 + \omega]^{-1}\cos\pi n x + \frac{k}{\lambda} L_2((\sigma_y)''_{xx})$$

The desired coefficients a_n and b_n can be determined from the system of equations (16), (17)

$$a_n = D_1 A_{1,n} + D_2 A_{2,n}, \quad b_n = D_1 B_{1,n} + D_2 B_{2,n}, \quad n \geqslant 1$$
$$a_0 = D_1 A_{1,0} + D_2 A_{2,0} - q/\omega, \quad b_0 = D_1 B_{1,0} + D_2 B_{2,0} \qquad (18)$$

where $A_{1,n}$, $B_{1,n}$ $(n \geqslant 1)$ is the solution of the system for $D_1 = 1$ and $D_2 = 0$. Similarly, $A_{2,n}$, $B_{2,n}$ is the solution for $D_1 = 0$ and $D_2 = 1$.

The undefined constants D_1 and D_2 can be found from Eq. (12), and the second boundary condition in (6), where σ_x is given by (8). In particular, the last equation from which to determine D_1 and D_2 has the form

$$F(1)h^3/12-\mu L_3(\sigma_y)-kL_4((\sigma_y)''_{xx})=0$$

$$L_3(\cdot)=\int_0^h y(\cdot)dy-\tfrac{1}{2}h\int_0^h(\cdot)dy$$

$$L_4(\cdot)=\int_0^h y\int_0^y\int_0^\eta(\cdot)d\xi d\eta dy-\tfrac{1}{2}h\int_0^h\int_0^y\int_0^\eta(\cdot)d\xi d\eta dy$$

(L_3 and L_4 are operators acting on the y coordinate).

In the case when the ends of the Bernoulli–Euler beam are fixed the constant C in (9) can be computed from the formula

$$C=\int_0^1\int_0^x\int_0^\xi R\,d\eta\,d\xi\,dx.$$

Thus, by computing a_n and b_n, one can determine σ_y and σ_{xy} from (13). The stress σ_x is written apart from a constant moment.

We shall determine the constant M from the condition

$$\int_0^h \sigma_x y\,dy=0$$

We obtain

$$M=6h^{-3}\sum_{n=1}^{\infty}a_n\int_0^h Z_n(y)y\,dy\cos\pi n \tag{19}$$

Note that, in general, the stresses can be represented in two ways. For example, σ_x can be computed either from (13) and (19) or from (8), taking the first relation in (13) and formula (14) into account. By comparing these two solutions, one can estimate the number N of terms that must be retained in the Fourier series.

Once the strains in the anisotropic layer are determined, one can find the displacements if the boundary condition $W_0=0$ is taken into account for $x=\pm 1$.

In Table 1 we list the stresses at certain characteristic points of the anisotropic layer for $h=s_0=0.01$, $\mu=4.5$, $k=4$, $E_x/E_0=0.5$, and $\lambda_0=40$. The stresses are given relative to the parameter q The stress σ_x was computed from (13) and (19), and σ_x^* was computed from (8). By comparing these two solutions, one can estimate the number of terms that must be kept in the Fourier series. In the given example $N=400$.

TABLE 1

x	$\sigma_y(0,x)$	$\sigma_{xy}(\frac{1}{2}h,x)\times 10^{-2}$	$\sigma_x(0,x)\times 10^{-4}$	$\sigma_x^*(0,x)\times 10^{-4}$
0	–0,665	0	0,348	0,350
58/96	–0,680	0,299	–0,016	–0,015
62/96	–0,598	0,321	–0,068	–0,066
72/96	–0,032	0,430	–0,220	–0,219
76/96	–0,744	0,476	–0,297	–0,296
84/96	–10,20	0,051	–0,418	–0,417
88/96	–15,82	–0,731	–0,367	–0,366
92/96	–3,722	–1,443	–0,175	–0,174
1	58,66	0	0,004	0,002

The numerical results obtained reveal that very substantial tensile stresses σ_y may arise at the ends $x=\pm 1$ of the layer as the compliance coefficient λ_0 decreases (cf. Table 1). The shear stresses near the free surface $x=\pm 1$ are also large. Thus, when computing the strength of a multi-layer construction consisting of stiff (anisotropic) and soft layers, it is advisable to determine the shear and normal stresses in the stiff layers.

The axial stresses σ_x vary linearly across the thickness of the beam. This provides a qualitative confirmation of the applicability of the fundamental assumptions of the theory of thin shells, for example, the Kirchhoff–Love hypothesis.

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